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ON THE COUPLING BETWEEN SPINNING PARTICLES AND COSMOLOGICAL GRAVITATIONAL WAVES

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The influence of spin in a system of classical particles on the propagation of gravitational waves is analyzed in the cosmological context of primordial thermal equilibrium. On a flat Friedmann-Robertson-Walker metric, when the precession is neglected, there is no contribution due to the spin to the distribution function of the particles. Adding a small tensor perturbation to the background metric, we study if a coupling between gravitational waves and spin exists that can modify the evolution of the distribution function, leading to new terms in the anisotropic stress, and then to a new source for gravitational waves. In the chosen gauge, the final result is that, in the absence of other kind of perturbations, there is no coupling between spin and gravitational waves.

Keywords: Gravitational Waves; spinning particle; Cosmology

1. Papapetrou Equations in the Isotropic and Homogenous Flat Universe

We study the influence of spinning particles on propagation of gravitational waves over a cosmological background. The equations of motion for a spinning particle in the framework of general relativity were derived by Papapetrou in 1951. He used

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a multipole expansion for the energy-momentum tensor of the particle such that, at dipole order, a deviation from geodesic motion and an equation describing spin precession are obtained.¹⁻² These equations are

$$\frac{D}{Ds}p^\mu = -\frac{1}{2}R^\mu_{\nu\rho\sigma}S^{\rho\sigma}u^\nu, \quad (1)$$

$$\frac{D}{Ds}S^{\mu\nu} = p^\mu u^\nu - p^\nu u^\mu, \quad (2)$$

where ds is the affine parameter, the vector p^μ is the generalized momentum, the antisymmetric tensor $S^{\mu\nu}$ is the angular momentum (spin) and $u^\mu = dx^\mu/ds$. In order to close the system we need to impose a supplementary condition that determines the center of mass of the spinning particle; we choose the Papapetrou condition:

$$S^{i0} = 0. \quad (3)$$

Other common choices that can be found in the literature are the Pirani condition ($S^{\mu\nu}u_\nu = 0$) and the Tulczyjew condition ($S^{\mu\nu}p_\nu = 0$), the former coinciding with the Papapetrou condition in the rest frame of the particle.

In our work we consider a flat Friedmann-Robertson-Walker (FRW) background described by the metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (4)$$

and assume that precession is absent³, so that Eq. (2) becomes:

$$\frac{D}{Ds}S^{\mu\nu} = 0, \quad (5)$$

and the generalized momentum coincides with the standard one.

Solving the Papapetrou equations in this case we obtain the time dependence of the angular momentum tensor, in terms of the scale factor:

$$S^{ij} = \frac{1}{a^2}\Sigma^{ij}, \quad (6)$$

where Σ^{ij} does not depend on time. For what concerns the momentum 4-vector, the equation of motion results being the standard geodesic equation, due to the symmetries of the homogeneous and isotropic FRW metric.

2. Boltzmann Equation: Influence of Spin-metric Terms

The Boltzmann equation for a non-collisional system of particles, when the spin is included through Eq. (1), is

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{p^i}{p^0} + \frac{\partial f}{\partial p_i} \frac{p^j p^k}{p^0} \frac{\partial g_{jk}}{\partial x^i} - \frac{1}{2p^0} \frac{\partial f}{\partial p_i} \left(\frac{1}{2} R_{i\alpha\beta\gamma} p^\alpha S^{\beta\gamma} \right) = 0, \quad (7)$$

where $f(x^i, p_j, t)$ is the distribution function of the fluid and p^i and p^0 are expressed

in terms of the independent variable p_i through the metric tensor. Defining $p = \sqrt{p_i p_i}$ so that $\partial p / \partial p_i = \hat{p}_i$ (hats denotes unit vectors) we can see that in the case of the FRW metric (4), the coupling term is equal to zero, because of the vanishing of S^{i0} and of the antisymmetry of Riemann tensor:

$$-\frac{1}{2p^0} \frac{\partial f}{\partial p^i} R_{i\alpha\beta\gamma} p^\alpha S^{\beta\gamma} = -\frac{1}{2p^0} \frac{\partial f}{\partial p} (R_{i0\beta\gamma} p^0 \hat{p}_i S^{\beta\gamma} + R_{ij\beta\gamma} p^j \hat{p}_i S^{\beta\gamma}). \quad (8)$$

Then we conclude that on an unperturbed FRW background the spin does not influence the evolution of the distribution function.

3. Spin-Gravitational Waves Coupling

The next step is to consider small tensor perturbations of the metric and see if in this case the presence of the spin can influence the time dependence of the distribution function, through the analysis of the perturbed Boltzmann equation. The solution will give the first order variation of the distribution function from which we can compute the anisotropic stress tensor acting as source in the differential equations for the tensor metric perturbation.

Let us consider a perturbation of the spatial metric:

$$g_{ij} = a^2 [\delta_{ij} + h_{ij}(\vec{x}, t)], \quad (9)$$

while the other components are left unperturbed: $g_{00} = -1$ and $g_{0i} = 0$. This amounts to choosing the synchronous gauge to study the evolution of perturbations. Here $h_{ij}(\vec{x}, t)$ is a small perturbation that satisfies the following conditions:

$$h_{ii} = 0, \quad \partial_i h_{ij} = 0. \quad (10)$$

i.e. it is traceless and divergenceless and thus represents a gravitational wave propagating over the FRW background. Considering the variations of the metric, the distribution function and the spin tensor, in the Fourier space the Boltzmann equation at first order is

$$\dot{\delta f} + i \hat{p}_i \hat{k}_i \delta f + \frac{ip}{2} \frac{\partial f^{(0)}}{\partial p} \hat{p}_i \hat{p}_j \hat{p}_k \hat{k}_i (h_{jk}(u) - h_{jk}(0)) - \frac{1}{2} \frac{\partial f^{(0)}}{\partial p} \frac{a}{k} \delta R_{i0lk} \hat{p}_i S^{lk} = 0, \quad (11)$$

where dot denotes derivative with respect to the time variable $u \equiv k \int_{t_1}^t dt' / a(t')$ and hats denote unit vectors; $f^{(0)}$ is the zeroth-order part of $f(\vec{x}, \vec{p}, t_1)$, which we assumed to have the usual ideal gas form. In computing this formula we have used the previous results about the unperturbed quantities.

The variation of the Riemann tensor in synchronous gauge is

$$\delta R_{i0lk} = \frac{ka}{2} (\dot{h}_{il,k} - \dot{h}_{ik,l}) \quad (12)$$

so that using Eq. (6) we can write Eq. (11) in the form

$$\dot{\delta f} + i \hat{p}_i \hat{k}_i \delta f + \frac{ip}{2} \frac{\partial f^{(0)}}{\partial p} \hat{p}_i \hat{p}_j \hat{p}_k \hat{k}_i (h_{jk}(u) - h_{jk}(0)) - \frac{i}{2} \hat{p}_i \frac{\partial f^{(0)}}{\partial p} \Sigma^{lj} k_j \dot{h}_{il} = 0. \quad (13)$$

This is a first order differential equation that enables us to write a formal solution for δf through integration with respect to u . We use this solution to compute the anisotropic stress part π_{ij} of the energy-momentum tensor T_j^i .

$$\pi_{ij} = T_j^i - \bar{p}\delta_{ij} \quad (14)$$

$$T_j^i = \frac{1}{\sqrt{-g}} \int f(x^i, p_j, t) \frac{p^i p_j}{p^0} d^3 p \quad (15)$$

with \bar{p} being the unperturbed pressure.

The final result is the sum of two different contributions

$$\pi_{ij}(u) = \pi_{ij}^{(0)}(u) + \pi_{ij}^{(S)}(u), \quad (16)$$

the first term is spin-independent^{4–5} and the second is the one we are interested in. We find

$$\pi_{ij}^{(S)} = \frac{i}{2} n \int_0^u du' K(u - u') k_m \Sigma^{lm} (k_i \dot{h}_{jl} + k_j \dot{h}_{il}) \quad (17)$$

where the integral kernel $K(u - u')(s) = \frac{1}{64} \int_{-1}^1 e^{ixs} (1 - x^2) x^2 dx$ and n is the particle number density. If we choose the z axis to coincide with the direction of propagation of the gravitational wave, we have that $\vec{k} = (0, 0, 1)$ and consequently, from the divergenceless of h_{ij} , that $h_{i3} = 0$. Then it is easy to see that the only components of $\pi_{ij}^{(S)}$ that are different from zero are those with one index equal to 3. The evolution equation for gravitational waves is

$$\ddot{h}_{ij} + 2\frac{\dot{a}}{a}\dot{h}_{ij} + h_{ij} = 16\pi G\pi_{ij}. \quad (18)$$

We can immediately see that, if only tensor modes (for which $h_{i3} = 0$) are present, the non-vanishing components of the spin dependent part of the anisotropic stress do not enter in the equation. Then there is no coupling between the gravitational waves and the spin of the particles in the fluid.

The next step in this work will be to study of supplementary conditions other than the Papapetrou condition, and the inclusion of scalar and tensor modes in the analysis. The existence of a coupling could be interesting in the perspective of the detection of cosmological gravitational waves by interferometers.

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